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Three-dimensional quadratic algebras: some realizations and representations

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Abstract
Four classes of three-dimensional quadratic algebra of the type $[Q_0, Q_\pm] = \pm Q_\pm$, $[Q_+, Q_-] = aQ_0^2 + bQ_0 + c$, where (a, b, c) are constants or central elements of the algebra, are constructed using a generalization of the well known two-mode bosonic realizations of $su(2)$ and $su(1,1)$. The resulting matrix representations and single variable differential operator realizations are obtained. Some remarks on the mathematical and physical relevance of such algebras are given.

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1. Introduction

In recent times there has been a great deal of interest in nonlinear deformations of Lie algebras because of their significant applications in several branches of physics. This is largely based on the realization that the physical operators relevant for defining the dynamical algebra of a system need not be closed under a linear (Lie) algebra, but might obey a nonlinear algebra. Such nonlinear algebras are, in general, characterized by commutation relations of the form

$$[T_i, T_j] = C_{ij}(\{T_k\}) \tag{1.1}$$

where the functions $\{C_{ij}\}$ of the generators $\{T_k\}$ are constrained by the Jacobi identity

$$[T_i, C_{jk}] + [T_j, C_{ki}] + [T_k, C_{ij}] = 0. \tag{1.2}$$

The functions $\{C_{ij}\}$ can be an infinite power series in $\{T_k\}$ as in the case of quantum algebras (with further Hopf algebraic restrictions) and q-oscillator algebras. When $\{C_{ij}\}$ are polynomials of the generators one obtains the so-called polynomially nonlinear, or simply polynomial, algebras. A special case of interest is when the commutation relations (1.1) take the form

$$[T_i, T_j] = c_{ij}^k T_k \qquad [T_i, T_\alpha] = t_{i\alpha}^\beta T_\beta \qquad [T_\alpha, T_\beta] = f_{\alpha\beta}(\{T_k\}) \tag{1.3}$$

containing a linear subalgebra. The simplest examples of such algebras occur when one gets a three-dimensional algebra

$$[N_0, N_\pm] = \pm N_\pm \qquad [N_+, N_-] = f(N_0) \tag{1.4}$$

in which $f(N_0)$ is a polynomial in N_0. In general,

$$\mathcal{C} = N_+N_- + g(N_0 - 1) = N_-N_+ + g(N_0) \tag{1.5}$$

is a Casimir operator of the algebra (1.4) where $g(N_0)$ can be determined from the relation

$$g(N_0) - g(N_0 - 1) = f(N_0). \tag{1.6}$$

If $f(N_0)$ is quadratic in N_0 we have a quadratic algebra, and if $f(N_0)$ is cubic in N_0 we have a cubic algebra. The polynomial algebras, in particular the quadratic and cubic algebras, and their representations are involved in the studies of several problems in quantum mechanics, statistical physics, field theory, Yang–Mills-type gauge theories, two-dimensional integrable systems, etc [1–16]. These algebras have also been found to occur in quantum optics with the observation that quantum optical Hamiltonians describing multiphoton processes have dynamical algebras described by polynomially deformed $su(2)$ and $su(1,1)$ algebras [17]. Coherent states of different kinds of nonlinear oscillator algebras have been presented by several authors [18–20]. Recently, we have presented a general unified approach for finding the coherent states of three-dimensional polynomial algebras [21]. The particular examples of three-dimensional quadratic algebras, which we have considered in [21], correspond to two of the four classes of algebras to be presented in this paper. Algebras of the type (1.4) have also been studied from a purely mathematical point of view [22] suggesting a rich theory of representations for them. Algebras of the type (1.4) with commutator $[N_+, N_-]$ replaced by the anticommutator $\{N_+, N_-\}$ leading to polynomial deformations of the superalgebra $osp(1|2)$ have also been investigated [23].

As is well known, in the case of classical Lie algebras, bosonic realizations play a very useful role in the representation theory and applications to physical problems. The main purpose of this paper is to study some aspects of three-dimensional quadratic algebras relating to bosonic realizations and the associated matrix representations, differential operator realizations, and physical relevance. In section 2 we briefly review the well-known construction of $su(2)$ and $su(1,1)$ algebras in terms of two-mode bosonic operators. In section 3 we show that a generalization of the Jordan–Schwinger method, using $su(2)$ or $su(1,1)$ and a boson algebra as the building blocks, leads to the construction of four classes of three-dimensional quadratic algebras. In sections 4–7 we exhibit the three-mode bosonic realizations of these quadratic algebras and derive the associated matrix representations and single variable differential operator realizations. Finally, in section 8 we conclude with a few remarks on the physical and mathematical relevance of these quadratic algebras.

2. Two-mode bosonic construction of $su(2)$ and $su(1,1)$: a brief review

Let us briefly recall the study of $su(2)$ and $su(1,1)$ in terms of two-mode bosonic realizations, to fix the framework and notations for our work. We let $(a_1, a_1^\dagger)$ and $(a_2, a_2^\dagger)$ be two mutually commuting boson annihilation–creation operator pairs, and $H_1 = N_1 + \frac{1}{2} = a_1^\dagger a_1 + \frac{1}{2}$ and $H_2 = N_2 + \frac{1}{2} = a_2^\dagger a_2 + \frac{1}{2}$. As is well known, (J_0, J_+, J_-) defined by

$$J_0 = \tfrac{1}{2}(H_1 - H_2) \qquad J_+ = a_1^\dagger a_2 \qquad J_- = J_+^\dagger = a_1 a_2^\dagger \tag{2.1}$$

satisfy the $su(2)$ algebra,

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0. \tag{2.2}$$

In this Jordan–Schwinger realization of $su(2)$, $H_1 + H_2$ is seen to be a central element: if

$$\mathcal{L} = \tfrac{1}{2}(H_1 + H_2) \tag{2.3}$$

then,

$$\left[\mathcal{L}, J_{0,\pm}\right] = 0. \tag{2.4}$$

The usual Casimir operator is

$$\mathcal{C} = J^2 = J_+J_- + J_0(J_0 - 1) = \mathcal{L}^2 - \tfrac{1}{4}. \tag{2.5}$$

Consequently, the application of the realization (2.1) on a set of $2j + 1$ two-mode Fock states $|n_1\rangle|n_2\rangle$, with constant $n_1 + n_2 = 2j$, leads to the $(2j + 1)$-dimensional unitary irreducible representation for each $j = 0, 1/2, 1, \ldots$. Thus, with $\{|j, m\rangle = |j + m\rangle|j - m\rangle \,|m = j, j - 1, \ldots, -j\}$ as the basis states, one gets the jth unitary irreducible representation

$$\begin{aligned}
&J_0|j, m\rangle = m|j, m\rangle \\
&J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle \\
&\mathcal{L}|j, m\rangle = \left(j + \tfrac{1}{2}\right)|j, m\rangle \qquad J^2|j, m\rangle = j(j + 1)|j, m\rangle \\
&m = j, j - 1, \ldots, -j.
\end{aligned} \tag{2.6}$$

Let us now consider the single variable differential operator realization corresponding to the above matrix representation (2.6). With the Fock–Bargmann correspondence

$$\left(a^\dagger, a\right) \longrightarrow \left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right) \qquad |n\rangle \longrightarrow \frac{z^n}{\sqrt{n!}} \tag{2.7}$$

we can make the association

$$\begin{aligned}
&|j, m\rangle \longrightarrow \frac{z_1^{j+m} z_2^{j-m}}{\sqrt{(j + m)!(j - m)!}} = \frac{z_2^{2j}(z_1/z_2)^{j+m}}{\sqrt{(j + m)!(j - m)!}} \\
&m = -j, -j + 1, \ldots, j - 1, j.
\end{aligned} \tag{2.8}$$

Since j is a constant for a given representation we can rewrite the above as a mapping to monomials

$$|j, m\rangle \longrightarrow \psi_{j,n}(z) = \frac{z^n}{\sqrt{n!(2j - n)!}} \qquad n = 0, 1, 2, \ldots, 2j. \tag{2.9}$$

Then, it is obvious that the above set of $(2j + 1)$ monomials (2.9) forms the basis carrying the finite-dimensional representation (2.6) corresponding to the single variable realization

$$J_0 = z\frac{\mathrm{d}}{\mathrm{d}z} - j \qquad J_+ = -z^2\frac{\mathrm{d}}{\mathrm{d}z} + 2jz \qquad J_- = \frac{\mathrm{d}}{\mathrm{d}z}. \tag{2.10}$$

In an analogous way, (K_0, K_+, K_-) defined by

$$K_0 = \tfrac{1}{2}(H_1 + H_2) \qquad K_+ = a_1^\dagger a_2^\dagger \qquad K_- = K_+^\dagger = a_1a_2 \tag{2.11}$$

satisfy the $su(1, 1)$ algebra

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0. \tag{2.12}$$

Now,

$$\mathcal{L} = \tfrac{1}{2}(H_1 - H_2) \tag{2.13}$$

is a central element of the algebra:

$$[\mathcal{L}, K_{0,\pm}] = 0. \tag{2.14}$$

The usual Casimir operator is

$$\mathcal{C} = K^2 = K_+K_- - K_0(K_0 - 1) = \tfrac{1}{4} - \mathcal{L}^2. \tag{2.15}$$

Consequently, the application of the realization (2.11) on any infinite set of two-mode Fock states $\{|k,n\rangle = |n+2k-1\rangle|n\rangle \; |n = 0,1,2,\ldots\}$, with constant $n_1 - n_2 = 2k-1$, leads to the infinite-dimensional unitary irreducible representation, the so-called positive discrete representation $\mathcal{D}^+(k)$, corresponding to any $k = 1/2, 1, 3/2, \ldots$.:

$$\begin{aligned}
&K_0|k,n\rangle = (k+n)|k,n\rangle \\
&K_+|k,n\rangle = \sqrt{(2k+n)(n+1)}\,|k,n+1\rangle \\
&K_-|k,n\rangle = \sqrt{(2k+n-1)n}\,|k,n-1\rangle \\
&\mathcal{L}|k,n\rangle = \left(k-\tfrac{1}{2}\right)|k,n\rangle \qquad K^2|k,n\rangle = k(1-k)|k,n\rangle \\
&n = 0,1,2,\ldots.
\end{aligned} \tag{2.16}$$

Note that the choice of basis states as $\{|k,n\rangle = |n\rangle|n+2k-1\rangle \; |n = 0,1,2,\ldots\}$, with $n_1-n_2 = 1-2k$, is also possible leading to the same representation (2.16), with $K^2 = k(1-k)$, but corresponding to $\mathcal{L} = \frac{1}{2} - k$.

As in the $su(2)$ case, we can make the association

$$|k,n\rangle \longrightarrow \frac{z_1^{n+2k-1} z_2^n}{\sqrt{(n+2k-1)!n!}} = \frac{(z_1 z_2)^n z_1^{2k-1}}{\sqrt{(n+2k-1)!n!}}. \tag{2.17}$$

Then, with k being constant in a given representation, it is obvious that the infinite set of monomials

$$\phi_{k,n}(z) = \frac{z^n}{\sqrt{(n+2k-1)!n!}} \qquad n = 0,1,2,\ldots \tag{2.18}$$

forms the basis carrying the representation (2.16) corresponding to the single variable realization

$$K_0 = z\frac{\mathrm{d}}{\mathrm{d}z} + k \qquad K_+ = z \qquad K_- = z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2k\frac{\mathrm{d}}{\mathrm{d}z}. \tag{2.19}$$

3. Construction of four classes of three-dimensional quadratic algebras

A three-dimensional quadratic algebra is defined, in general, by the commutation relations

$$[Q_0, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = aQ_0^2 + bQ_0 + c \tag{3.1}$$

where the structure constants (a,b,c) are constants, or central elements of the algebra so that they take constant values in any irreducible representation. In general, a Casimir operator of the algebra (3.1) is

$$\mathcal{C} = Q_+Q_- + \tfrac{1}{3}aQ_0^3 - \tfrac{1}{2}(a-b)Q_0^2 + \tfrac{1}{6}(a-3b+6c)Q_0 - c. \tag{3.2}$$

When $a = c = 0$ and $b = \pm 2$ one gets, respectively, $su(2)$ and $su(1,1)$ as special cases.

Quadratic algebras and their representations, physically relevant to the situations, have been studied in physics literature in the context of several specific physical problems where (a,b,c) take particular values depending on the problem studied. But there seems to be no systematic study of realizations, representations, and any classification of the quadratic algebras corresponding to arbitrary values of (a,b,c). The main purpose of this paper is to observe that a generalization of the Jordan–Schwinger method leads to four classes of three-dimensional quadratic algebras, in each of which the constants (a,b,c) take a particular series of values. To this end we proceed as follows.

Let $\left\{\left(a_i, a_i^\dagger\right) \middle| i = 1,2,3\right\}$ be three mutually commuting boson annihilation–creation operator pairs. Let $N_i = a_i^\dagger a_i$ and $H_i = N_i + \frac{1}{2}$ for each $i = 1,2,3$. Now, we let

$$Q_0 = \tfrac{1}{2}(J_0 - N_3) \qquad Q_+ = J_+ a_3 \qquad Q_- = J_- a_3^\dagger \tag{3.3}$$

replacing in (2.1) $(H_1, a_1^\dagger, a_1)$ by (J_0, J_+, J_-) respectively, and $(H_2, a_2^\dagger, a_2)$ by $(N_3, a_3^\dagger, a_3)$ respectively (note that (J_0, J_+, J_-) already contain two sets of bosonic operators). It is now straightforward to observe that, with $\mathcal{L}$ and $\mathcal{J}$ defined by

$$\mathcal{L} = \tfrac{1}{2}(J_0 + N_3) \qquad \mathcal{J} = J_+J_- + J_0(J_0 - 1) = J^2 \tag{3.4}$$

we have

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm \\ &[Q_+, Q_-] = -3Q_0^2 - (2\mathcal{L} - 1)Q_0 + (\mathcal{J} + \mathcal{L}(\mathcal{L} + 1)) \end{aligned} \tag{3.5}$$

and

$$[\mathcal{L}, \mathcal{J}] = 0 \qquad [\mathcal{L}, Q_{0,\pm}] = 0 \qquad [\mathcal{J}, Q_{0,\pm}] = 0. \tag{3.6}$$

The Casimir operator (3.2) becomes

$$\mathcal{C} = Q_+Q_- - Q_0^3 - (\mathcal{L} - 2)Q_0^2 + \big(\mathcal{J} + \mathcal{L}^2 + 2\mathcal{L} - 1\big)Q_0 - (\mathcal{J} + \mathcal{L}(\mathcal{L} + 1)). \tag{3.7}$$

Next, let us replace in (2.11) $\big(H_1, a_1^\dagger, a_1\big)$ by (J_0, J_+, J_-) respectively, and $\big(H_2, a_2^\dagger, a_2\big)$ by $\big(N_3, a_3^\dagger, a_3\big)$ respectively. This leads to the definitions

$$Q_0 = \tfrac{1}{2}(J_0 + N_3) \qquad Q_+ = J_+a_3^\dagger \qquad Q_- = J_-a_3 \tag{3.8}$$

which obey the algebra

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm \\ &[Q_+, Q_-] = 3Q_0^2 + (2\mathcal{L} + 1)Q_0 - (\mathcal{J} + \mathcal{L}(\mathcal{L} - 1)) \end{aligned} \tag{3.9}$$

with

$$\mathcal{L} = \tfrac{1}{2}(J_0 - N_3) \qquad \mathcal{J} = J_+J_- + J_0(J_0 - 1) = J^2 \tag{3.10}$$

being the central elements of the algebra. The Casimir operator becomes

$$\mathcal{C} = Q_+Q_- + Q_0^3 + (\mathcal{L} - 1)Q_0^2 - \big(\mathcal{J} + \mathcal{L}^2\big)Q_0 + (\mathcal{J} + \mathcal{L}(\mathcal{L} - 1)). \tag{3.11}$$

Let us now use the $su(1,1)$ generators instead of the $su(2)$ generators in the above scheme. Thus we replace in (3.3) (J_0, J_+, J_-) by (K_0, K_+, K_-) respectively. The result is that

$$Q_0 = \tfrac{1}{2}(K_0 - N_3) \qquad Q_+ = K_+a_3 \qquad Q_- = K_-a_3^\dagger \tag{3.12}$$

satisfy the algebra

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm \\ &[Q_+, Q_-] = 3Q_0^2 + (2\mathcal{L} - 1)Q_0 + (\mathcal{K} - \mathcal{L}(\mathcal{L} + 1)) \end{aligned} \tag{3.13}$$

with

$$\mathcal{L} = \tfrac{1}{2}(K_0 + N_3) \qquad \mathcal{K} = K_+K_- - K_0(K_0 - 1) = K^2 \tag{3.14}$$

as the central elements. The Casimir operator becomes

$$\mathcal{C} = Q_+Q_- + Q_0^3 + (\mathcal{L} - 2)Q_0^2 + \big(\mathcal{K} - \mathcal{L}^2 - 2\mathcal{L} + 1\big)Q_0 - (\mathcal{K} - \mathcal{L}(\mathcal{L} + 1)). \tag{3.15}$$

If, in (3.8), we replace (J_0, J_+, J_-) by (K_0, K_+, K_-) respectively, the result is that

$$Q_0 = \tfrac{1}{2}(K_0 + N_3) \qquad Q_+ = K_+a_3^\dagger \qquad Q_- = K_-a_3 \tag{3.16}$$

satisfy the algebra

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm \\ &[Q_+, Q_-] = -3Q_0^2 - (2\mathcal{L} + 1)Q_0 - (\mathcal{K} - \mathcal{L}(\mathcal{L} - 1)) \end{aligned} \tag{3.17}$$

with

$$\mathcal{L} = \tfrac{1}{2}(K_0 - N_3) \qquad \mathcal{K} = K_+K_- - K_0(K_0 - 1) = K^2 \tag{3.18}$$

being the central elements. The Casimir operator is

$$\mathcal{C} = Q_+Q_- - Q_0^3 - (\mathcal{L}-1)Q_0^2 - \left(\mathcal{K}-\mathcal{L}^2\right)Q_0 + (\mathcal{K}-\mathcal{L}(\mathcal{L}-1)). \tag{3.19}$$

Thus, by combining the generators of $su(2)$, or $su(1,1)$, and an oscillator algebra, analogous to the way two oscillator algebras are combined to get $su(2)$ or $su(1,1)$, we get four classes of three-dimensional quadratic algebras of the type (3.1) with (a, b, c) as central elements. Let us call the algebras (3.5), (3.9), (3.13) and (3.17), respectively, $Q^-(2)$, $Q^+(2)$, $Q^-(1,1)$ and $Q^+(1,1)$ where the superscripts $\pm$ indicate whether the corresponding Q_0 is a sum or a difference (correspondingly the forms of $Q_\pm$ get fixed). We shall study the realizations and representations of these quadratic algebras in the following sections.

4. Representations of $Q^-(2)$

For the algebra $Q^-(2)$ defined in (3.5) $\mathcal{J}$ and $\mathcal{L}$ are to be constants in any irreducible representation. This implies that we can take the basis states of the irreducible representations as

$$|j,l,m\rangle = |j,m\rangle|2l-m\rangle \tag{4.1}$$

such that

$$\mathcal{J}|j,l,m\rangle = j(j+1)|j,l,m\rangle \qquad \mathcal{L}|j,l,m\rangle = l|j,l,m\rangle \tag{4.2}$$

where $\{|j,m\rangle\}$ are the $su(2)$ basis states (2.6) and $\{|2l-m\rangle\}$ are the oscillator eigenstates such that $N_3|2l-m\rangle = (2l-m)|2l-m\rangle$. This requires $2l-m$ to be a non-negative integer for all values of m. Thus, two cases arise which are to be considered separately. In general, the Casimir operator (3.7) has the value $(l+1)[j(j+1)-l(l-1)]$ in both cases.

Case 1. $2l - j \geqslant 0$. In this case the set of basis states

$$|j,l,m\rangle = |j,m\rangle|2l-m\rangle \qquad m = -j, -j+1, \ldots, j-1, j \tag{4.3}$$

carry the (j,l)th, $(2j+1)$-dimensional, unitary irreducible representation

$$\begin{aligned}
&Q_0|j,l,m\rangle = (m-l)|j,l,m\rangle \\
&Q_+|j,l,m\rangle = \sqrt{(j-m)(j+m+1)(2l-m)}|j,l,m+1\rangle \\
&Q_-|j,l,m\rangle = \sqrt{(j+m)(j-m+1)(2l-m+1)}|j,l,m-1\rangle \\
&m = -j, -j+1, \ldots, j-1, j.
\end{aligned} \tag{4.4}$$

Let us look at the two-dimensional representations. These correspond to $j = 1/2$ and for each value of $l = 1/4, 3/4, 5/4, \ldots,$ there is a two-dimensional representation given by

$$\begin{aligned}
&Q_0 = \begin{pmatrix} -\frac{1}{2}-l & 0 \\ 0 & \frac{1}{2}-l \end{pmatrix} \\
&Q_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2l+\frac{1}{2}} & 0 \end{pmatrix} \qquad Q_- = \begin{pmatrix} 0 & \sqrt{2l+\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \\
&\mathcal{J} = 3/4 \qquad \mathcal{L} = l \qquad \mathcal{C} = (-4l^3+7l+3)/4
\end{aligned} \tag{4.5}$$

as can be verified directly.

Case 2. $2l - j < 0$. In this case the set of basis states

$$|j, l, m\rangle = |j, m\rangle|2l - m\rangle \qquad m = -j, -j+1, \ldots, 2l \tag{4.6}$$

carry the (j, l)th unitary irreducible representation of dimension $j + 2l + 1$ given by

$$\begin{aligned} &Q_0|j, l, m\rangle = (m - l)|j, l, m\rangle \\ &Q_+|j, l, m\rangle = \sqrt{(j - m)(j + m + 1)(2l - m)}|j, l, m + 1\rangle \\ &Q_-|j, l, m\rangle = \sqrt{(j + m)(j - m + 1)(2l - m + 1)}|j, l, m - 1\rangle \\ &m = -j, -j + 1, \ldots, 2l. \end{aligned} \tag{4.7}$$

In this case for any $j > 1/2$ there is a two-dimensional representation corresponding to $l = (1 - j)/2$. Explicitly,

$$\begin{aligned} &Q_0 = \tfrac{1}{2}\begin{pmatrix} -j-1 & 0 \\ 0 & 1-j \end{pmatrix} \\ &Q_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2j} & 0 \end{pmatrix} \qquad Q_- = \begin{pmatrix} 0 & \sqrt{2j} \\ 0 & 0 \end{pmatrix} \\ &\mathcal{J} = j(j+1) \qquad \mathcal{L} = (1-j)/2 \qquad \mathcal{C} = (-3j^3 + 5j^2 + 11j + 3)/8 \end{aligned} \tag{4.8}$$

as can be verified directly.

By using the two-mode bosonic realization of $su(2)$ we can write down the three-mode bosonic realization of $Q^-(2)$ as

$$Q_0 = \tfrac{1}{4}\left(a_1^\dagger a_1 - a_2^\dagger a_2 - 2a_3^\dagger a_3\right) \qquad Q_+ = a_1^\dagger a_2 a_3 \qquad Q_- = a_1 a_2^\dagger a_3^\dagger. \tag{4.9}$$

Correspondingly we can take the basis states (4.1) of the irreducible representations as the three-mode Fock states

$$|j, l, m\rangle = |j + m\rangle|j - m\rangle|2l - m\rangle. \tag{4.10}$$

Then the action of (Q_0, Q_+, Q_-) defined by (4.9) on these basis states leads to the irreducible representations (4.4) and (4.7), respectively, in the two cases $2l - j \geqslant 0$ and $2l - j < 0$.

Let us now make the association

$$|j, m, l\rangle \longrightarrow \frac{z_2^{2j} z_3^{2l+j}(z_1/z_2 z_3)^{j+m}}{\sqrt{(j+m)!(j-m)!(2l-m)!}}. \tag{4.11}$$

Since j and l are constants, this shows that the set of functions

$$\psi^-_{j,l,n}(z) = \frac{z^n}{\sqrt{n!(2j-n)!(2l+j-n)!}} \qquad n = 0, 1, 2, \ldots, 2j \text{ or } 2l + j \tag{4.12}$$

forms the basis for the representation (4.4) or (4.7), respectively, corresponding to the single variable realization

$$\begin{aligned} &Q_0 = z\frac{\mathrm{d}}{\mathrm{d}z} - j - l \\ &Q_+ = z^3\frac{\mathrm{d}^2}{\mathrm{d}z^2} - (2l + 3j + 1)z^2\frac{\mathrm{d}}{\mathrm{d}z} + 2j(2l - j)z \qquad Q_- = \frac{\mathrm{d}}{\mathrm{d}z}. \end{aligned} \tag{4.13}$$

5. Representations of $Q^+(2)$

For the algebra $Q^+(2)$ defined in (3.9) the constancy of $\mathcal{J}$ and $\mathcal{L}$ in any irreducible representation implies that the basis states can be taken as

$$|j, l, m\rangle = |j, m\rangle|m - 2l\rangle \tag{5.1}$$

such that

$$\mathcal{J}|j,l,m\rangle = j(j+1)|j,l,m\rangle \qquad \mathcal{L}|j,l,m\rangle = l|j,l,m\rangle \tag{5.2}$$

where $\{|j,m\rangle\}$ are the $su(2)$ basis states and $\{|m-2l\rangle\}$ are the oscillator eigenstates such that $N_3|m-2l\rangle = (m-2l)|m-2l\rangle$. The requirement that $m-2l$ has to be a non-negative integer for any value of m implies that $2l+j$ always has to be zero or a negative integer. Thus the irreducible representations of this algebra are labelled by the pairs $\{(j,l)\}$ such that $2l+j$ is zero or a negative integer. The corresponding (j,l)th irreducible representation is always $(2j+1)$-dimensional and is given by

$$\begin{aligned}
&Q_0|j,l,m\rangle = (m-l)|j,l,m\rangle \\
&Q_+|j,l,m\rangle = \sqrt{(j-m)(j+m+1)(m-2l+1)}|j,l,m+1\rangle \\
&Q_-|j,l,m\rangle = \sqrt{(j+m)(j-m+1)(m-2l)}|j,l,m-1\rangle \\
&m = -j, -j+1, \ldots, j-1, j.
\end{aligned} \tag{5.3}$$

The Casimir operator (3.11) takes the value $(1-l)[j(j+1)-l(l+1)]$ in this representation. Note that the irreducible representations in this case are analogous to those of $su(2)$ except that there are infinitely many inequivalent irreducible representations of the same dimension corresponding to the infinity of the possible values of l for each value of j.

The two-dimensional representations correspond to $j = 1/2$ and $l = -1/4, -3/4, -5/4, \ldots$, and are given by

$$\begin{aligned}
&Q_0 = \begin{pmatrix} -\frac{1}{2}-l & 0 \\ 0 & \frac{1}{2}-l \end{pmatrix} \\
&Q_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{1}{2}-2l} & 0 \end{pmatrix} \qquad Q_- = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}-2l} \\ 0 & 0 \end{pmatrix} \\
&\mathcal{J} = 3/4 \qquad \mathcal{L} = l \qquad \mathcal{C} = \tfrac{1}{4}\left(4l^3 - 7l + 3\right).
\end{aligned} \tag{5.4}$$

By using the two-mode bosonic realization of $su(2)$ we can write down the three-mode bosonic realization of $Q^+(2)$ as

$$Q_0 = \tfrac{1}{4}\left(a_1^\dagger a_1 - a_2^\dagger a_2 + 2a_3^\dagger a_3\right) \qquad Q_+ = a_1^\dagger a_2 a_3^\dagger \qquad Q_- = a_1 a_2^\dagger a_3. \tag{5.5}$$

Correspondingly we can take the basis states (5.1) of the irreducible representations as the three-mode Fock states

$$|j,l,m\rangle = |j+m\rangle|j-m\rangle|m-2l\rangle. \tag{5.6}$$

Then the action of (Q_0, Q_+, Q_-) defined by (5.5) on these basis states leads to the irreducible representation (5.3).

Now, we can make the association

$$|j,m,l\rangle \longrightarrow \frac{z_2^{2j} z_3^{-(2l+j)} (z_1 z_3/z_2)^{j+m}}{\sqrt{(j+m)!(j-m)!(m-2l)!}}. \tag{5.7}$$

Since j and l are constants, this shows that the set of functions

$$\psi^+_{j,l,n}(z) = \frac{z^n}{\sqrt{n!(2j-n)!(n-2l-j)!}} \qquad n = 0, 1, 2, \ldots, 2j \tag{5.8}$$

forms the basis for the representation (5.3) corresponding to the single variable realization

$$Q_0 = z\frac{\mathrm{d}}{\mathrm{d}z} - j - l \qquad Q_+ = -z^2\frac{\mathrm{d}}{\mathrm{d}z} + 2jz \qquad Q_- = z\frac{\mathrm{d}^2}{\mathrm{d}z^2} - (2l-j-1)\frac{\mathrm{d}}{\mathrm{d}z}. \tag{5.9}$$

6. Representations of $Q^-(1,1)$

For the algebra $Q^-(1,1)$ defined in (3.13) the condition that $\mathcal{K}$ and $\mathcal{L}$ take constant values in an irreducible representation fixes the basis to be the set of states

$$|k,l,n\rangle = |k,n\rangle|2l-k-n\rangle \qquad n = 0,1,2,\ldots,(2l-k) \tag{6.1}$$

where $\{|k,n\rangle\}$ are the $su(1,1)$ basis states (2.16), $\{|2l-k-n\rangle\}$ are the oscillator states such that $N_3|2l-k-n\rangle = (2l-k-n)|2l-k-n\rangle$,

$$2l-k = 0,1,2,\ldots \qquad k = 1/2, 1, 3/2, \ldots \tag{6.2}$$

and

$$\mathcal{K}|k,l,n\rangle = k(1-k)|k,l,n\rangle \qquad \mathcal{L}|k,l,n\rangle = l|k,l,n\rangle. \tag{6.3}$$

The basis states (6.1) carry the (k,l)th, $(2l-k+1)$-dimensional, unitary irreducible representation:

$$\begin{aligned}
&Q_0|k,l,n\rangle = (k-l+n)|k,l,n\rangle \\
&Q_+|k,l,n\rangle = \sqrt{(n+2k)(n+1)(2l-k-n)}\,|k,l,n+1\rangle \\
&Q_-|k,l,n\rangle = \sqrt{(n+2k-1)n(2l-k-n+1)}\,|k,l,n-1\rangle \\
&n = 0,1,2,\ldots,(2l-k).
\end{aligned} \tag{6.4}$$

The Casimir operator (3.15) has the value $(l+1)[k(1-k)+l(l-1)]$ in this representation.

For this algebra there is a two-dimensional representation for each value of $k = 1/2, 1, 3/2, \ldots$, as given by

$$\begin{aligned}
&Q_0 = \tfrac{1}{2}\begin{pmatrix} k-1 & 0 \\ 0 & k+1 \end{pmatrix} \qquad Q_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2k} & 0 \end{pmatrix} \qquad Q_- = \begin{pmatrix} 0 & \sqrt{2k} \\ 0 & 0 \end{pmatrix} \\
&\mathcal{K} = k(1-k) \qquad \mathcal{L} = l = \tfrac{1}{2}(k+1) \qquad \mathcal{C} = \tfrac{1}{8}\big(-3k^3 - 5k^2 + 11k - 3\big)
\end{aligned} \tag{6.5}$$

as can be verified directly.

By using the two-mode bosonic realization of $su(1,1)$ we can write down the three-mode bosonic realization of $Q^-(1,1)$ as

$$Q_0 = \tfrac{1}{4}\big(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3 + 1\big) \qquad Q_+ = a_1^\dagger a_2^\dagger a_3 \qquad Q_- = Q_+^\dagger = a_1 a_2 a_3^\dagger. \tag{6.6}$$

Translating the basis states (6.1) into the three-mode Fock states it is found that the action of (Q_0, Q_+, Q_-) defined by (6.6) on the basis states $\{|k,l,n\rangle = |n+2k-1\rangle|n\rangle|2l-k-n\rangle\}$ leads to the representation (6.4).

As before, let us make the association

$$|k,l,n\rangle \longrightarrow \frac{z_1^{2k-1} z_3^{2l-k} (z_1 z_2/z_3)^n}{\sqrt{(n+2k-1)!\,n!\,(2l-k-n)!}}. \tag{6.7}$$

Since k and l are constants for a given representation we can take

$$\phi^-_{k,l,n}(z) = \frac{z^n}{\sqrt{(n+2k-1)!\,n!\,(2l-k-n)!}} \qquad n = 0,1,2,\ldots,(2l-k) \tag{6.8}$$

as the set of basis functions for the single variable realization

$$Q_0 = z\frac{\mathrm{d}}{\mathrm{d}z} + k - l \qquad Q_+ = -z^2\frac{\mathrm{d}}{\mathrm{d}z} + (2l-k)z \qquad Q_- = z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2k\frac{\mathrm{d}}{\mathrm{d}z} \tag{6.9}$$

leading to the representation (6.4).

7. Representations of $Q^+(1,1)$

For the algebra $Q^+(1,1)$ defined in (3.17) the constancy of $\mathcal{K}$ and $\mathcal{L}$ in any irreducible representation fixes the basis to be the set of states

$$|k,l,n\rangle = |k,n\rangle|n+k-2l\rangle \qquad n = 0,1,2,\ldots \tag{7.1}$$

where $\{|k,n\rangle\}$ are the $su(1,1)$ basis states, $\{|n+k-2l\rangle\}$ are the oscillator states such that $N_3|n+k-2l\rangle = (n+k-2l)|n+k-2l\rangle$,

$$k-2l = 0,1,2,\ldots \qquad k = 1/2, 1, 3/2, \ldots \tag{7.2}$$

and

$$\mathcal{K}|k,l,n\rangle = k(1-k)|k,l,n\rangle \qquad \mathcal{L}|k,l,n\rangle = l|k,l,n\rangle. \tag{7.3}$$

The set of basis states (7.1) carry the (k,l)th, infinite-dimensional unitary irreducible representation:

$$\begin{aligned} &Q_0|k,l,n\rangle = (k-l+n)|k,l,n\rangle \\ &Q_+|k,l,n\rangle = \sqrt{(n+2k)(n+1)(n+k-2l+1)}\,|k,l,n+1\rangle \\ &Q_-|k,l,n\rangle = \sqrt{(n+2k-1)n(n+k-2l)}\,|k,l,n-1\rangle \\ &n = 0,1,2,\ldots. \end{aligned} \tag{7.4}$$

The Casimir operator (3.19) has the value $l(l-k^2)$ in this representation.

In terms of three bosonic modes the realization of (Q_0, Q_+, Q_-) is given by

$$Q_0 = \tfrac{1}{4}\big(a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_3^\dagger a_3 + 1\big) \qquad Q_+ = a_1^\dagger a_2^\dagger a_3^\dagger \qquad Q_- = Q_+^\dagger = a_1 a_2 a_3. \tag{7.5}$$

The application of this realization to the set of three-mode Fock states

$$|k,l,n\rangle = |n+2k-1\rangle|n\rangle|n+k-2l\rangle \qquad n = 0,1,2,\ldots \tag{7.6}$$

leads to the (k,l)th irreducible representation (7.4).

From the association

$$|k,l,n\rangle \longrightarrow \frac{z_1^{2k-1} z_3^{k-2l}(z_1 z_2 z_3)^n}{\sqrt{(n+2k-1)!\,n!\,(n+k-2l)!}} \tag{7.7}$$

it is clear that we can take

$$\phi^+_{k,l,n}(z) = \frac{z^n}{\sqrt{(n+2k-1)!\,n!\,(n+k-2l)!}} \tag{7.8}$$

as the set of basis functions for the single variable realization associated with the representation (7.4). The corresponding realization is

$$\begin{aligned} &Q_0 = z\frac{\mathrm{d}}{\mathrm{d}z} + k - l \qquad Q_+ = z \\ &Q_- = z^2\frac{\mathrm{d}^3}{\mathrm{d}z^3} + (3k-2l+2)z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \big(2k^2 - 4kl + 2k\big)\frac{\mathrm{d}}{\mathrm{d}z}. \end{aligned} \tag{7.9}$$

8. Concluding remarks

As already mentioned in the introduction, polynomial algebras occur in several physical problems and have mathematically rich structures. Here we would like to make some observations with reference to the three-dimensional quadratic algebras we have constructed.

Let us first look at the Dicke model in quantum optics. This model, generalizing the Jaynes–Cummings model, describes the interaction of the radiation field with a collection of identical two-level atoms located within a distance much smaller than the wavelength of the radiation. In the particular case when the atoms interact resonantly with a single mode coherent cavity field, the (Tavis–Cummings) Hamiltonian, under the electric dipole and rotating wave approximations, is given by (in the units $\hbar = 1$)

$$H = \omega\big(J_0 + a^\dagger a\big) + gJ_+a + g^*J_-a^\dagger \tag{8.1}$$

where ω is the frequency of the field mode (and the atomic transition), $J_0 + a^\dagger a$ is the excitation number operator, and g is the coupling constant. The annihilation and creation operators a and $a^\dagger$ correspond to the single mode radiation field. The operators

$$J_0 = \sum_j \sigma_0^j \qquad J_\pm = \sum_j \sigma_\pm^j \tag{8.2}$$

with $\sigma_{0,\pm}^j$ as mutually commuting triplets of the Pauli matrices, obey the $su(2)$ algebra and define the collective atomic operators. From purely physical arguments it is possible to construct the matrix representation of the Hamiltonian (8.1) and to study its spectrum and applications [35, 36] (see [37] for an extensive review). The excitation number operator (corresponding to our $\mathcal{L}$) and J^2 are known integrals of motion for the system. Thus it is now easy to recognize $Q^-(2)$ as the dynamical algebra of the Hamiltonian (8.1). The two cases of representations (4.4)–(4.7) we have found are exactly the ones identified in the literature from a physical point of view. We hope that this precise identification of the dynamical algebra of the Dicke model will help its further understanding. In this regard, our earlier proposal of a general method for constructing the Barut–Girardello-type and Perelomov-type coherent states of any three-dimensional polynomial algebra [21] should be useful. In [21] we have, in particular, considered the examples of $Q^\pm(1, 1)$ and the cubic Higgs algebra with reference to the construction of coherent states.

One should also note the following. As in the case of any Lie algebra one may also present a polynomial algebra equivalently by choosing linear combinations of the generators as new generators. For example, following [38], we may take $X_3 = Q_0 + \mathcal{L}$, $X_+ = J_+a$ and $X_- = J_-a^\dagger$ as the three generators of $Q^-(2)$. Then the corresponding algebraic relations are

$$[X_3, X_\pm] = \pm X_\pm \qquad [X_+, X_-] = -3X_3^2 + (4\mathcal{L} + 1)X_3 + J^2. \tag{8.3}$$

This is easily seen by substituting $Q_0 = X_3 - \mathcal{L}$ in (3.5). In [38] the three-dimensional quadratic algebra defined by (8.3), equivalent to $Q^-(2)$, has been considered for constructing the relative-phase operator for the Dicke model.

Next let us consider the Hamiltonian

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_3^\dagger a_3 + 2 \tag{8.4}$$

which describes, in the units $\hbar = 1$ and $\omega = 1$, a three-dimensional anisotropic quantum harmonic oscillator with the frequency in the third direction twice that in the perpendicular plane. If we relate H to Q_0 in (7.5) it is clear that $Q^+(1, 1)$ is the dynamical algebra of the system (8.4). From the representations (7.4) one can easily arrive at the result that the spectrum of H is the set of all integers $\geqslant 2$. Let us look at the invariance algebra of H. From the construction of $Q^-(1, 1)$ we recognize that

$$H = 4\mathcal{L} + 1 \tag{8.5}$$

where $\mathcal{L}$ is a central element of the algebra generated by $(Q_0, Q_\pm)$ in (3.13) or (6.6). Thus, $(\mathcal{K}, Q_0, Q_\pm)$ are the integrals of motion for the system (8.4), or in other words, $Q^-(1, 1)$ is the invariance algebra of the system. Since $\mathcal{L}$ has the spectrum

$$\mathcal{L} = l = n/4 \qquad n = 1, 2, 3, \ldots \tag{8.6}$$

it is clear that the Hamiltonian (8.4) has the spectrum

$$H = N + 2 \qquad N = 0, 1, 2, \ldots. \tag{8.7}$$

Each level can be labelled by the eigenvalues of a complete set of commuting operators $(H-2, \mathcal{K})$. It is interesting to compute the degeneracy of the Nth level using the representation theory of the algebra (3.13). For the Nth level the value of $\mathcal{L}$ is $l = (N+1)/4$. Calculating the corresponding values of k for which finite-dimensional representations are possible we find that the dimensions of the associated irreducible representations are $(1, 2, \ldots, 2m+1)$ if $N = 4m$ or $4m+1$, and $(1, 2, \ldots, 2m+2)$ if $N = 4m+2$ or $4m+3$. The degeneracy of the level is the sum of the dimension of the $k = 1/2$ representation and twice the dimensions of $k > 1/2$ representations. One has to count the dimensions of $k > 1/2$ representations twice in the sum since there are two possible choices for the bases leading to the same representation in these cases as already noted. Now, the four cases, $N = 4m, 4m+1, 4m+2$ and $4m+3$, are to be considered separately. The result is as follows: the degeneracies of the levels, $N = 4m, 4m+1, 4m+2$ and $4m+3$, respectively, are $(2m+1)^2$, $(2m+1)(2m+2)$, $4(m+1)^2$ and $2(m+1)(2m+3)$. In other words, the number of compositions of the integer N (partitions with ordering taken into account) in the prescribed pattern $n_1+n_2+2n_3$, with the interchange of n_1 and n_2 taken into account, is $(2m+1)^2$, $(2m+1)(2m+2)$, $4(m+1)^2$ and $2(m+1)(2m+3)$, if $N = 4m, 4m+1, 4m+2$, and $4m+3$, respectively. It is to be noted that in this example the sum of all the dimensions of the irreducible representations associated with the given $l = (N+1)/4$ gives the number of partitions of N in the pattern $n_1 + n_2 + 2n_3$, disregarding the interchange of n_1 and n_2. This leads to the result that the number of such partitions is $(m+1)(2m+1)$ for $N = 4m$ or $4m+1$ and $(m+1)(2m+3)$ for $N = 4m+2$ or $4m+3$. Thus, it is interesting to observe this connection between a three-dimensional quadratic algebra and the theory of partitions.

It should be noted that, if one can identify a given three-dimensional quadratic algebra as belonging to one of the four classes we have considered, then its representation theory can be worked out immediately, at least partially. For example, we observe that

$$Q_0 = a^\dagger a \qquad Q_+ = \frac{1}{\sqrt{3}}\left(a^\dagger\right)^3 \qquad Q_- = \frac{1}{\sqrt{3}}\, a^3 \tag{8.8}$$

obey the algebra

$$[Q_0, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = -3Q_0^2 - 3Q_0 + 2. \tag{8.9}$$

This algebra is uniquely identified with $Q^+(1, 1)$ with $\mathcal{L} = l = 1$ and $\mathcal{K} = k(1-k) = -2$ (or $k = 2$). Correspondingly the algebra is seen to have the infinite-dimensional representation given by

$$\begin{aligned} &Q_0|n\rangle = (n+1)|n\rangle \\ &Q_+|n\rangle = (n+1)\sqrt{n+4}|n+1\rangle \qquad Q_-|n\rangle = n\sqrt{n+3}|n-1\rangle \\ &n = 0, 1, 2, \ldots. \end{aligned} \tag{8.10}$$

The fact that the representations we have discussed are not complete is clear from the following example. For a two-dimensional anisotropic quantum harmonic oscillator the Hamiltonian is

$$H = a_1^\dagger a_1 + 2a_2^\dagger a_2 + \tfrac{3}{2} \tag{8.11}$$

in the units $\hbar = 1$ and $\omega = 1$. The invariance algebra of this Hamiltonian is generated by

$$Q_0 = \tfrac{1}{4}\left(a_1^\dagger a_1 - 2a_2^\dagger a_2 + \tfrac{1}{2}\right) \qquad Q_+ = \tfrac{1}{2}\left(a_1^\dagger\right)^2 a_2 \qquad Q_- = \tfrac{1}{2} a_1^2 a_2^\dagger \tag{8.12}$$

and the algebra is given by

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm \\ &[Q_+, Q_-] = 3Q_0^2 + \tfrac{1}{2}(H-3)Q_0 - \tfrac{1}{16}H(H+2) + \tfrac{3}{8}. \end{aligned} \tag{8.13}$$

This algebra can be readily identified with $Q^-(1,1)$ corresponding to $\mathcal{L} = (H-1)/4$ and $\mathcal{K} = k(1-k) = 3/16$ or $k = 1/4$ or $3/4$. It may be noted that the corresponding representations cannot be presented in terms of the three-boson Fock states since these, considered in section 6, correspond only to $k = 1/2, 1, 3/2, \ldots$. To get the representations of the algebra (8.13) one will have to combine the representations of a boson algebra with the representations of $su(1,1)$ for $k = 1/4$ or $3/4$ (in terms of single boson Fock states). A detailed discussion of the algebraic approach to the two-dimensional quantum system of an anisotropic oscillator with an additional singular potential in one direction is found in [13].

An interesting possibility is suggested by the structure of the algebra $Q^-(1,1)$. Let us define

$$N = Q_0 \qquad A = \frac{1}{\sqrt{\mathcal{L}(\mathcal{L}+1)-\mathcal{K}}}\,Q_- \qquad A^\dagger = \frac{1}{\sqrt{\mathcal{L}(\mathcal{L}+1)-\mathcal{K}}}\,Q_+. \tag{8.14}$$

Then the algebra (3.13) becomes

$$\begin{aligned}&[N,A] = -A \qquad \left[N,A^\dagger\right] = A^\dagger\\ &\left[A,A^\dagger\right] = 1 - \frac{2\mathcal{L}-1}{\mathcal{L}(\mathcal{L}+1)-\mathcal{K}}\,N - \frac{3}{\mathcal{L}(\mathcal{L}+1)-\mathcal{K}}\,N^2.\end{aligned} \tag{8.15}$$

We may consider this as the defining algebra of a quadratic oscillator, corresponding to a special case of the general class of deformed oscillators [24–34]:

$$[N,A] = -A \qquad \left[N,A^\dagger\right] = A^\dagger \qquad \left[A,A^\dagger\right] = F(N). \tag{8.16}$$

The quadratic oscillator (8.15) belongs to the class of generalized deformed parafermions [12]. It should be interesting to study the physics of assemblies of quadratic oscillators. In fact, the canonical fermion, with

$$N = \begin{pmatrix}0&0\\0&1\end{pmatrix} \qquad f = \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad f^\dagger = \begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{8.17}$$

is a quadratic oscillator! We observe that

$$[N,f] = -f \qquad \left[N,f^\dagger\right] = f^\dagger \qquad \left[f,f^\dagger\right] = 1 - \tfrac{1}{2}N - \tfrac{3}{2}N^2. \tag{8.18}$$

To conclude, starting with $su(2)$, $su(1,1)$, and an oscillator algebra, we have constructed four classes of three-dimensional quadratic algebras of the type

$$[Q_0, Q_\pm] = \pm Q_\pm \qquad [Q_+, Q_-] = aQ_0^2 + bQ_0 + c. \tag{8.19}$$

In each class the structure constants (a,b,c) take a particular series of values. We have also found for these algebras the three-mode bosonic realizations, corresponding matrix representations, and single variable differential operator realizations. We hope that this work will lead to further insights into a more complete understanding of the representation theory of arbitrary three-dimensional quadratic algebras.

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